Group Theory 16

12 November 2023 09:08

- If G is a finite group and has a unique Sylow p-subgroup for each prime divisor p of $|G|$ then G is the direct product of its Sylow groups

- If $|G| = p^n$ where p is a prime and if $0 \le k \le n$ then G contains a normal subgroup of order p^k.

- If p is a prime then every group G of order $2p$ is either cyclic or dihedral

Theorem :- If $|G| = pq$ where $p > q$ are primes Then either G is cyclic or $G = \langle a, b \rangle$ where $a^q = 1$, $b^p = 1$, $aba^{-1} = b^m$ and $m^q \equiv 1 \pmod p$ and $m \not\equiv 1 \pmod p$. If $q \nmid (p-1)$ then the second case cannot occur.

- If $p > q$ are primes then every group G of order pq contains a normal subgroup of order p. If $q \nmid (p-1)$ then G is cyclic

Q> Suppose that a group G has a subgroup of order n. Prove that the intersection of all subgroups of G of order n is a normal subgroup of G.

Ans:- H be the intersection of all such subgroups of G of order n. $g \in G$, let $g^{-1}Hg$ is not in $H \Rightarrow g^{-1}Hg$ is not in some

Ans:- H be the [illegible]

$g \in G$, let $g^{-1}Hg$ is not in $H \Rightarrow g^{-1}Hg$ is not in some subgroup of order n, say K.

Then $g^{-1}Kg$ is also a subgroup of G of order n.

$\Rightarrow H \subset g^{-1}Kg$

$\Rightarrow H = g^{-1}K'g$ for some $K' \leq K$

$\Rightarrow K' = gHg^{-1}$ for some $g \in G$

$\subset K$

Hence contradiction.

Hence $g^{-1}Hg = H \Rightarrow H$ is normal of order n.

Q> Let H be a subgroup of a group G and let $g \in G$. Prove that $D = g^{-1}Hg$ is a subgroup of G. Also, if $Ord(H) = n$ then $Ord(g^{-1}Hg) = Ord(H) = n$

Ans:- $a, b \in D$ $\quad h_1, h_2 \in H \Rightarrow h_1^{-1}h_2 \in H$

$a = g^{-1}h_1 g$ $\quad a^{-1}b = (g^{-1}h_1 g)^{-1}(g^{-1}h_2 g)$

$b = g^{-1}h_2 g$ $\quad = g^{-1}h_1^{-1}g g^{-1}h_2 g$

$= g^{-1}h_1^{-1}h_2 g = g^{-1}h_3 g \quad h_3 \in H$

$e \in D$

$\Rightarrow a^{-1}b \in D$

$Ord(H) = n$

$g^{-1}h_1 g = g^{-1}h_2 g \Leftrightarrow h_1 = h_2$

$\Rightarrow Ord(H) = Ord(g^{-1}Hg) = n$

Q> Let G be a group of order 30 Prove that G has an element of order 15.

Ans:- Think about it.

Theorem:-

Let n be the number of Sylow p-Subgroups of a finite group G. Then $n \mid Ord(G)$ and $p \mid (n-1)$.

Q> Let G be a noncyclic group of order 57 Prove that G has exactly 38 elements of order 3. $57 = 19 \times 3$

Ans:- There will be only one Sylow 19-Subgroup.

$a \in G$ then $Ord(a) \mid Ord(G) \Rightarrow a = 1, 3, 19$

$a \neq e \Rightarrow Ord(a) = 3, 19$

Sylow 19-subgroup contains Ord of 19 elements

$|\text{Sylow } 19\text{-subgroup}| = 19 \Rightarrow a \neq e$ elements are 18.

So there are $57 - 18 - \underset{a=e}{1} = 38$ elements of order 3

Q> Let G be a group of order 100 Prove that G has a normal subgroup of order 50.

Ans:- G has subgroup K of order 2

G will also have a normal subgroup H of order 25.

Then HK is a subgroup of G.

$\gcd(2, 25) = 1 \Rightarrow Ord(HK) = 50$

$\Rightarrow [G : HK] = 2 \Rightarrow HK$ is normal subgroup of order 50